Mathematical Appendix

This is a formal exposition analyzing the effects of cuts in marginal tax rates within a standard Keynesian model underlying the IS-LM analysis commonly taught in undergraduate economics courses. This model seems to underlie current perceptions of the effects of tax cuts.

We specify a consumption function C, depending on the level of income Y and interest rates r: C(Y,r). C_1 and C_2 and are the partial derivatives of C with respect to Y and r, respectively. Standard assumptions are that $1 > C_1 > 0$ and $C_2 < 0$: higher income induces higher consumption with a marginal propensity to consume less than 1, while higher levels of interest rates induce lower consumption.

Similarly, we specify a demand for money function L(Y,r), where $L_1 > 0$ and $L_2 < 0$. Higher levels of income induce higher demand for money, while higher levels of interest rates induce lower demand for money. Finally, investment demand is specified as I(r), where the effect of r on I is negative: $I' < 0$.

Again, all these specifications are standard. Our only modification will be to make the consumption and money-demand functions depend on after-tax income instead of total income and on after-tax interest rates instead of pre-tax rates. We consider both lump-sum taxes T and marginal taxes tY, where t is the marginal tax rate, so that total taxes paid are R=T+tY, and after-tax income is (1-t)Y-T, while after-tax interest rates are (1-t)r, since lump-sum taxes have no effect on interest income or costs.

So, $C \equiv C[(1-t)Y-T, (1-t)r]$, and, $L \equiv L[(1-t)Y-T, (1-t)r]$, with signs of C_1, C_2, L_1, and L_2 as before. All we have done is acknowledge that individuals' behavior depends on after-tax magnitudes. It is standard to specify after-tax income within the consumption function. It is less common to specify after-tax income as affecting money demand or after-tax interest rates as affecting consumption or money demand, but that is because these analyses most commonly analyze changes in lump-sum taxes.

Certainly, individuals' spending and asset-accumulation decisions depend on the after-tax flows and after-tax yields available to them, and this is what we reflect in our specifications for C and L. As we'll see, the difference in the two types of taxation results in dramatic differences in the effect on income and interest rates. As for why investment depend on pre-tax interest rates, we discuss that at the close.

Under these specifications, standard analysis specifies IS and LM curves as sets of points (Y,r) for which:

IS Curve: $C[(1-t)Y-T, (1-t)r] + I(r) + G - Y = 0$

and

LM Curve: $L[(1-t)Y-T, (1-t)r] - M = 0.$

G is the level of government purchases of goods and services, while M is the level of the money stock. As we are concerned with fiscal policy here, we won't analyze the effects of changes in M, but we will analyze the effects of changes in G because of its role as a potential funding source for tax rate cuts.

Equilibrium levels of Y and r are those where both the IS and LM specifications are satisfied. We can shock these equilibrium conditions with changes in T, t, and G to determine the effects on the equilibrium levels of Y and r. For incremental changes in these, the following simultaneous equation system holds:

$$\begin{bmatrix} (1-t)C_1 - 1 & (1-t)C_2 + I' \\ (1-t)L_1 & (1-t)L_2 \end{bmatrix} \begin{bmatrix} dY \\ dr \end{bmatrix} = \begin{bmatrix} C_1 dT + C_1 Y dt + rC_2 dt - dG \\ L_1 dT + L_1 Y dt + rL_2 dt + dM \end{bmatrix} \quad (1)$$

We'll call the matrix on the left D. Then the determinant

$$|D| \equiv (1-t)\{L_2[(1-t)C_1-1] - L_1[(1-t)C_2+I']\} > 0, \quad (2)$$

given the specifications stated above: $1 > C_1 > 0$, $C_2 < 0$, $L_1 > 0$, $L_2 < 0$, and $I' < 0$.

Concerning changes in lump-sum taxes T: $\begin{bmatrix}\frac{\partial Y}{\partial T}\\ \frac{\partial r}{\partial T}\end{bmatrix} = D^{-1}\cdot\begin{bmatrix}C_1\\ L_1\end{bmatrix}$ and by Cramer’s Rule and (2),

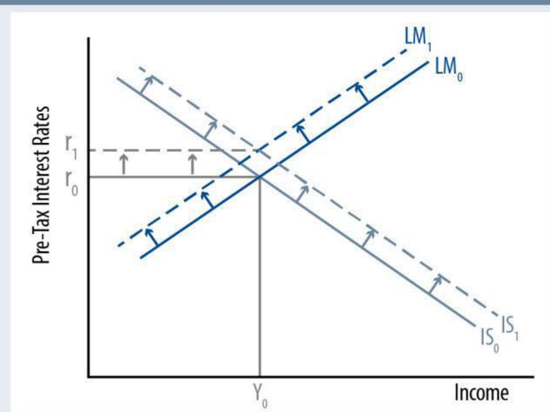
$$\frac{\partial Y}{\partial T} = \begin{vmatrix}C_1 & (1-t)C_2+I'\\ L_1 & (1-t)L_2\end{vmatrix}/|D| = \frac{L_2[(1-t)C_1]-L_1[(1-t)C_2+I']}{|D|} = \ldots = \frac{L_2}{|D|}+\frac{1}{1-t} \qquad (3)$$

and $$\frac{\partial r}{\partial T} = \begin{vmatrix}(1-t)C_1-1 & C_1\\ (1-t)L_1 & L_1\end{vmatrix}/|D| = -L_1/|D| < 0. \qquad (4)$$

The effect of T on r is clearly negative, which means that cuts in lump-sum taxes raise interest rates. It is interesting that in this treatment, the effect of tax cuts on income, $\partial Y/\partial T$, is uncertain. This is different from the standard result—that tax cuts raise income—because we have depicted money demand as depending in after-tax income. If the L_1dT term were zeroed out in the right side of (1), then $\partial Y/\partial T$ would clearly be negative, so that lump-sum tax cuts would unambiguously raise income.

Exhibit 1from the text illustrates these changes. With a decline in T, after-tax income is higher for any given level of Y, so that L is higher. So r then needs to be higher to satisfy the LM curve. In other words, a cut in lump-sum taxes causes the LM curve to shift up and to the left. Also, with a higher level of after-tax income due to the drop in T, C is higher, so that r needs to be higher to satisfy the IS condition. So, the IS curve shifts up and to the right.

Exhibit 1
Effects of Cuts in Lump-Sum Taxes



Source: Western Asset

Pre-tax interest rates are clearly higher, but whether total income Y rises or not depends on whether the IS or LM curve has shifted more. That is, it depends on the relative sizes of C_1, C_2, C_1, L_1, L_2, and I'.

Below, we’ll have reference to the term $1+t\cdot\partial Y/\partial T$. This is the extent to which government revenues change when lump-sum taxes are changed. The lump-sum tax change has a direct effect of 1 and then whatever effect there is on income further affects revenues by a factor of t. From (2) and (3), the change in revenues $R \equiv tY+T$ in response to a change in lump-sum taxes is

$$\frac{\partial R}{\partial T} = 1+t\cdot\frac{\partial Y}{\partial T} = 1+t\left(\frac{L_2}{|D|}+\frac{1}{1-t}\right) = \frac{1}{1-t} + t\cdot\frac{L_2}{|D|} = \frac{1}{|D|}\{L_2[(1-t)C_1-1] - L_1[(1-t)C_2+I'] + tCL_2\}$$

$$= \frac{1}{|D|}\{L_2(1-t)(C_1-1) - L_1[(1-t)C_2+I']\} \equiv B/|D| > 0. \qquad (5)$$

B is defined as the term in braces in (5). B is positive since each of its components are positive. So, regardless of the effect of T on Y, increases in T raise revenue, and vice versa.
Now, consider changes in marginal tax rates t:

$\begin{bmatrix}\frac{\partial Y}{\partial t}\\ \frac{\partial r}{\partial t}\end{bmatrix} = D^{-1}\cdot\begin{bmatrix}C_1Y+rC_2\\ L_1Y+rL_2\end{bmatrix}$. Again using Cramer’s Rule and simplifying,

$$\frac{\partial Y}{\partial t} = \begin{vmatrix}C_1Y+rC_2 & (1-t)C_2+I'\\ L_1Y+rL_2 & (1-t)L_2\end{vmatrix}/|D| = \ldots = Y\cdot\partial Y/\partial T - \frac{rI'L_2}{|D|}$$

$$= \frac{Y}{1-t} + \frac{YL_2}{|D|} - \frac{rI'L_2}{|D|} = \frac{Y}{1-t} + \frac{L_2}{|D|}(Y-rI'). \qquad (6)$$

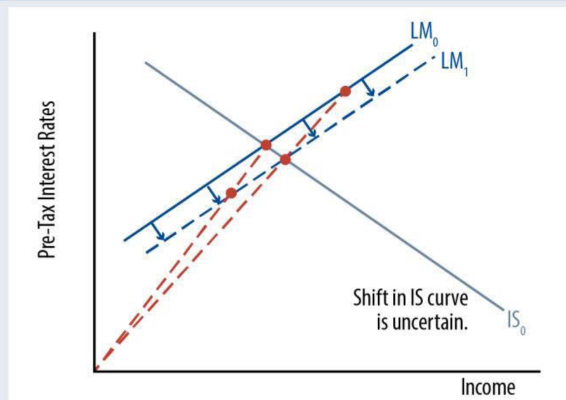
Similarly, $$\frac{\partial r}{\partial t} = \begin{vmatrix} (1-t)C_1 - 1 & C_1Y + rC_2 \\ (1-t)L_1 & L_1Y + rL_2 \end{vmatrix} / |D| = \ldots = \frac{r}{1-t} - \frac{L_1}{|D|}(Y - rI') . \qquad (7)$$

The quantities $Y/(1-t)$ and $r/(1-t)$ crop up a lot in this discussion. They state how much Y and r would have to change to keep after-tax magnitudes constant opposite a change in marginal tax rates. To see this, notice that $d[(1-t)Y] = -Ydt+(1-t)dY= 0$ if $dY = dt \cdot Y/(1-t)$. So, for a given change in tax rates, a change in Y of $Y/(1-t)$ times that magnitude keeps after-tax income constant. Similar reasoning holds for $r/(1-t)$.

So, for "unfinanced" cuts in marginal tax rates, the effects on both income and pre-tax interest rates are of uncertain sign. To see this terms of the IS-LM analysis, consider Exhibit 1A. If in response to a decline in t, Y and r change by equal proportions to return after-tax levels to their original values, then L will be unchanged, and the LM curve will still be in balance. In other words, each point on the LM curve moves downward along a ray from the origin.

As for the IS curve, hold Y and r at their original levels in response to a decline in t. Then I, G, and Y are unchanged. The rise in after-tax income works to increase C, while the rise in after-tax interest rates works to lower C. In other words, it is uncertain how the IS curve shifts in this context, which is why the effects of the tax rate cuts on Y and r are uncertain.

Exhibit 1A
Effects of Unfinanced Cuts in Marginal Tax Rates



Source: Western Asset

If Y and r were to decline in response to a cut in t so as to keep after-tax levels unchanged, then the LM curve would still be in balance, but the IS curve would be in a state where $C+I+G > Y$, since the decline in r increases I, while the decline in Y decreases C by less than Y declines (since $C_1 < 1$). Y and r would have to rise from there to restore balance in the IS curve, and they would have to rise in proportion $-L_2/L_1$, to ***sustain*** balance on the LM curve. ($-L_2/L_1$ is the slope of that LM curve.) This is what (6) and (7) say. In response to a cut in t, Y and r move by initial amounts $Y/(1-t)$ and $r/(1-t)$, with both then moving back by $(Y-rI')$, the disruption to the IS curve from this initial movement, in proportions $-L_2/L_1$.

Solving for the change in government revenues in response to a change in marginal tax rates yields:

$$\frac{\partial R}{\partial t} = Y + t\cdot\frac{\partial Y}{\partial t} = Y + t\left[\frac{Y}{1-t} + \frac{L_2}{|D|}(Y - rI')\right] = \frac{Y}{1-t} + \frac{tL_2}{|D|}(Y - rI'). \qquad (8)$$

Finally, the effects of changes in government purchases are: $\begin{bmatrix} \frac{\partial Y}{\partial G} \\ \frac{\partial r}{\partial G} \end{bmatrix} = D^{-1} \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. By Cramer's Rule,

$$\frac{\partial Y}{\partial G} = \begin{vmatrix} -1 & (1-t)C_2 + I' \\ 0 & (1-t)L_2 \end{vmatrix} / |D| = -(1-t)L_2/|D| = 1 - (1-t)\cdot \partial Y/\partial T > 0. \qquad (9)$$

$$\frac{\partial r}{\partial G} = \begin{vmatrix} (1-t)C_1 - 1 & -1 \\ (1-t)L_1 & 0 \end{vmatrix} / |D| = (1-t)L_1/|D| = -(1-t)\, \partial r/\partial T > 0. \qquad (10)$$

The last substitution in (9) follows from utilizing (3), while that in (10) follows from (4). Not surprisingly, the effects of government spending on income work much the same as do the effects of lump-sum tax cuts. (That is, the formulae for $\partial Y/\partial G$ and $\partial r/\partial G$ contains the terms $\partial Y/\partial T$ and $\partial r/\partial T$, respectively.) Unlike the case for lump-sum tax cuts, the effects of government purchases on income are clearly positive, since government purchases increase income directly whereas tax cuts work through the consumption function, and also since government purchases have no effect on the LM curve.

Finally, the effects of changes in government purchases on revenues are, from (5):

$$\frac{\partial R}{\partial G} = t\cdot\frac{\partial Y}{\partial G} = t[1 - (1-t)\cdot \partial Y/\partial T] \quad \text{and} \quad 1 - \frac{\partial R}{\partial G} = (1-t)\left\{1 + t\cdot\frac{\partial Y}{\partial T}\right\} = (1-t)B/|D|. \qquad (11)$$

Explicitly-Financed Tax Cuts. We can determine the effects of fully-financed tax rate cuts by combining cuts in marginal rates with either hikes in lump-sum taxes or cuts in government spending that leave the government budget balance unaffected. So, if marginal tax rates are cut and lump-sum taxes are raised enough to keep total government revenues unchanged, it must be that $dR = \frac{\partial R}{\partial t}dt + \frac{\partial R}{\partial T}dT = 0$,

and from (5) and (8),

$$\frac{\partial T}{\partial t}\Big|_{dR=0} = -Y + \frac{YrI'L_2}{B}. \qquad (12)$$

From (3), (5), (6), and (12),

$$\frac{\partial Y}{\partial t}\Big|_{dR=0} = \frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial T}\cdot\frac{\partial T}{\partial t}\Big|_{dR=0} = \ldots = -\frac{rI'L_2}{B(1-t)} < 0. \qquad (13)$$

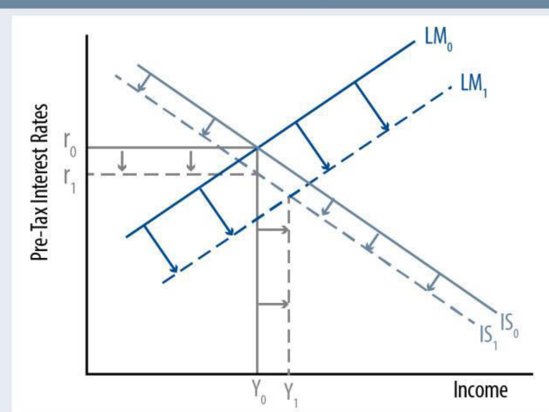
And by (4), (5), (7), and (12),

$$\frac{\partial r}{\partial t}\Big|_{dR=0} = \frac{\partial r}{\partial t} + \frac{\partial r}{\partial T}\cdot\frac{\partial T}{\partial t}\Big|_{dR=0} = \ldots = \frac{r}{1-t}\{L_2(1-t)(C_1-1) - L_1(1-t)C_2\}/B = \frac{r}{1-t}(1+I'L_1/B) > 0. \qquad (14)$$

(14) is positive since the term in pointed brackets is positive. That the ratio of that term to B is less 1 is shown by the last expression. So, ***cuts in marginal tax rates financed by lump-sum taxes raise income and lower pre-tax interest rates, but after-tax interest rates are higher on net***.

These results can be seen in terms of an IS-LM analysis as per Exhibit 2 from the text. With a financed cut in marginal rates, at the previous equilibrium levels of Y and r, after-tax income is unchanged, while after-tax interest rates are higher, so that L < M. To bring the LM curve back into balance, pre-tax interest rates must fall enough to bring after-tax interest rates to their previous level. So, the LM curve shifts down in parallel fashion by proportion r/(1-t).

Exhibit 2
Effects of Financed Cuts in Marginal Tax Rates



Source: Western Asset

For the IS curve, assume Y and r are at their previous levels, and a financed tax rate cut occurs. Then I, G, and Y are unchanged, and after-tax income is unchanged, but after-tax interest rates are higher, so C is lower, C+I+G < Y, and Y or r have to decline to bring the IS curve back in balance. So, the IS curve has to shift down and to the left. But keep Y unchanged and lower r enough to bring after-tax interest rates back to the pre-tax-cut level. Then C, G, and Y are unchanged, but r is lower, so I is higher and C+I+G > Y. So, r or Y have to be higher than this. In other words, the IS curve shifts downward less than does the LM curve, which is why Y must increase as a result of the shift. r is clearly lower since both curves have shifted downward.

Now, consider the effect of a cut in marginal tax rates financed by cuts in government purchases. Then, the government budget balance is B = R-G, and from (5), (8), and (11),

$$dB = \frac{\partial R}{\partial t}dt + \frac{\partial R}{\partial G}dG - dG = 0, \text{ so that } \frac{\partial G}{\partial t}|_{dB=0} = (\partial R/\partial t)/(1-\partial R/\partial G) = \frac{Y}{1-t} - \frac{t}{1-t}\frac{L_2 rI'}{B}. \quad (15)$$

So, from (6), (9), and (15), and from (7), (10), and (15),

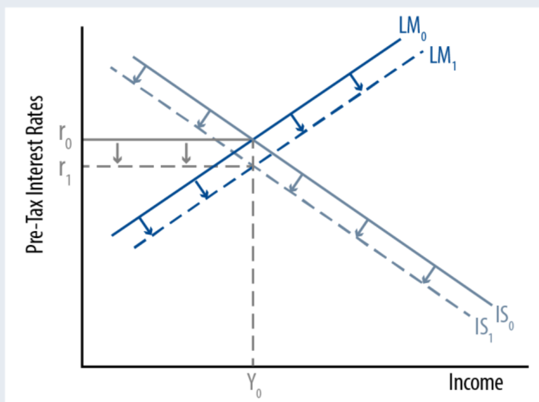
$$\frac{\partial Y}{\partial t}|_{dB=0} = \frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial G}\cdot\frac{\partial G}{\partial t}|_{dB=0} = \ldots = \frac{Y}{1-t} - \frac{rI'L_2}{(1-t)B}, \text{ and} \quad (16)$$

$$\frac{\partial r}{\partial t}|_{dB=0} = \frac{\partial r}{\partial t} + \frac{\partial r}{\partial G}\cdot\frac{\partial G}{\partial t}|_{dB=0} = \ldots = \frac{r}{1-t} + \frac{rI'L_1}{(1-t)B} = \frac{r}{1-t}(1+I'L_1/B) > 0. \quad (17)$$

The effect shown in (17) is identical to that in (14). ***For a given cut in marginal tax rates, interest rates drop by the same amount regardless of how the marginal tax rate cut is financed***. (13) and (16) differ by the amount Y/(1-t), so financing marginal tax rates cuts with cuts in government purchases has less stimulative effect on income. This is equivalent to saying that the balanced-budget multiplier is positive when government purchases are financed by lump-sum taxes. Since (16) is of uncertain sign, so is the balanced-budget multiplier for government purchases financed by marginal tax rates.

In terms of the IS-LM diagram (Exhibit 3), since spending cuts do not affect the money market, tax rate cuts financed by spending cuts move the LM curve down and to the right in exactly the same way as Exhibit 1A. As for the IS curve, for a given Y and r on that curve, suppose cuts in t are financed by a cut in G. At the original levels of Y and r, the higher level of after-tax income has an upward effect on C, but this is smaller than the decline in G, since C_1 <1 and since the decline in G matches the size of increase in after-tax income. Meanwhile, with r unchanged and t lower, after-tax interest rates are higher, which imparts a downward effect on consumption. On both scores, C+I+G < Y, so Y or r have to decline to restore balance on the IS curve. In other words, the IS curve shifts down and to the left.

Exhibit 3
Effects of Cuts in Marginal Tax Rates Financed by Cuts in Government Purchases



Source: Western Asset

The fact that effects of financed changes in t on r do not depend on the means of financing is an interesting finding. It means that changes in government purchases financed by changes in lump-sum taxes have no effect on interest rates, since moving from the changes in Exhibit 2 to that in Exhibit 3 merely involve equal decreases in G and in R. That this is true can be seen from (3), (4), (9), and (10) as follows.

Consider a change in G. This will induce some change in Y that will then change revenues by tY in the same direction as the change in G. So, T needs to change by proportion (1-t)Y to finance the change in G. This is just another way of saying that $\frac{\partial T}{\partial G}|_{dB=0} = (1-t)$. But (9) says that $\frac{\partial Y}{\partial G} + (1-t)\frac{\partial Y}{\partial T} = 1$. This magnitude measures the effect on Y when changes in G are financed by changes in T. Similarly, it is clear from (4) and (10) that $\frac{\partial r}{\partial G} + (1-t)\frac{\partial r}{\partial T} = 0$, and this is the net effect on r from changes in G financed by changes in T.

We can see this as well in terms of the IS-LM diagrams. With hikes in G financed by hikes in T, the resulting decline in after-tax income reduces L. In order to restore balance on the LM curve, Y would have to rise enough to restore after-tax income to its previous level. That is, the LM curve shifts down and to the right. For the IS curve, at previous levels of Y and r, Y and I are unchanged, but G is higher and C is lower by the decline in after-tax income multiplied by the marginal propensity to consumer, C_1. With C_1<1, C is lower than by less than G is higher, so that C+I+G>Y, so Y has to increase to restore balance: the IS curve shifts right.

Suppose r is unchanged, and Y has increased enough to restore (1-t)Y-T to its initial value. Then after-tax income and interest rates are unchanged, so the LM curve remains in balance. For the IS curve, C and I are unchanged, and Y has risen by exactly the increase in G, since we know that after-tax income is unchanged, so that taxes have risen by exactly the same amount as the increase in Y, which equals the increase in G.

In summary, it is clear in this model that the balanced-budget multiplier is exactly 1, when government purchases are financed directly by lump-sum taxes, with these operations having no effect on interest rates. Of course, as seen in (16) and (17), increases in government purchases financed by higher ***marginal*** tax rates raise interest rates and have an uncertain effect on income. The latter is obviously the relevant case for real-world exercises, but if, as in the text, hiking "lump-sum taxes" can be seen to be equivalent to cuts in tax preferences, the exercise analyzed in the last four paragraphs would have some real-world relevance.

We have treated investment I(r) as depending on pre-tax interest rates, because that is what is indicated by models of optimal capital accumulation. As the benefits of investment are taxable, while the costs of investment are tax-deductible, marginal tax rates have no direct effect on investment for a given level of pre-tax interest rates.
[*] Actually, though, if we model I as depending on (1-t)r rather than r, the math above is much simpler, while the results of the effects of marginal tax rate cuts on interest rates are much the same.[†]

Finally, while it might seem strange that cuts in marginal tax rates do not have more stimulative impact on income than we have found here, this merely reflects the limitations of the two-market ISLM Keynesian model analyzed here, where labor markets are assumed to be in excess supply, so that increases in the supply of labor induced by marginal tax rate cuts have no impact on the aggregate economy and where goods markets are assumed to be in excess supply, so tax cuts stimulate investment only to the extent they lower interest rates.

Once labor market equilibrium is considered in the analysis, the higher supply of labor due to tax rate cuts induces increases in income, increasing the downward impetus to interest rates. Similarly, a fuller model of business behavior would result in a more direct increase in investment activity in response to cuts in marginal tax rates. Of course, those changes would bring the analysis closer to a neoclassical growth model, and we have already mentioned in the text that in that framework, cuts in marginal tax rates clearly stimulate the economy and clearly lower pre-tax interest rates. What we have shown here is that even in a properly specified, simplistic Keynesian model, financed cuts in marginal tax rates can clearly be seen to induce lower interest rates, even as they stimulate the economy.

[*] Whether marginal tax rates affect the pre-tax level of interest is another story of course, in fact the subject of this whole paper. However, saying that changes in marginal tax rates have an effect on equilibrium pre-tax interest rates is not the same thing as saying the investors respond to after-tax rather than pre-tax interest rates. Again, for a given level of pre-tax rates, an optimal investor would accumulate capital up to the point where the pre-tax return on investment matches the pre-tax cost of capital/borrowing. So, I here should depend on r and not (1-t)r.

[†] Actually, with I depending on (1-t)r, r declines more in response to a cut in marginal tax rates than what we have found here. This is because the rise in after-tax interest rates now serves to reduce investment in addition to the other effects analyzed here. So, equilibrium r has to be yet lower to offset this effect.